Calculating angular diameter distances

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1 Introduction

In the research domain of gravitational lenses, one often has to supply angular diameter distances to perform specific calculations. Unfortunately, there is no way to measure such distances directly, but they have to be calculated from the observed redshifts of astronomical objects.

Below, an isotropic and homogeneous universe described by the Friedmann-Robertson-Walker (FRW) metric

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - a(t)^2\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right) \qquad (k = +1, 0 \text{ or } -1),$$

will be assumed to describe our universe well.

2 Angular diameter distances in a FRW metric

Figure 1 shows an observer in Euclidean space, looking at an object at distance D, perpendicular to the line of sight and subtending an angle $\Delta\theta$. When the angle is small, the following relation holds:

$$d \approx \Delta\theta D.$$

This is exactly the way an angular diameter distance in a general metric is defined: the size of the object (at the time the light we receive now was emitted) must equal the corresponding angle time the angular diameter distance.

In a general FRW metric, the situation we are interested in is depicted in figure 2. The coordinate system is chosen in such a way that the observer is at the origin, the object lies on a surface of constant ϕ and the radial coordinate of the endpoints is r. Note that the coordinates of the object being viewed are fixed.

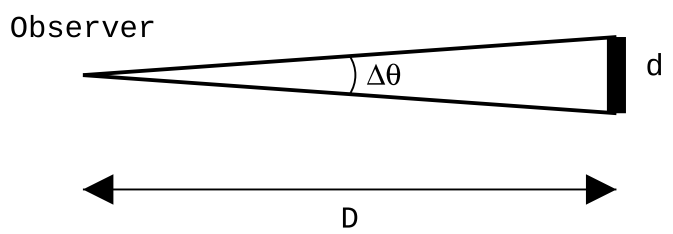

Figure 1: Euclidean space

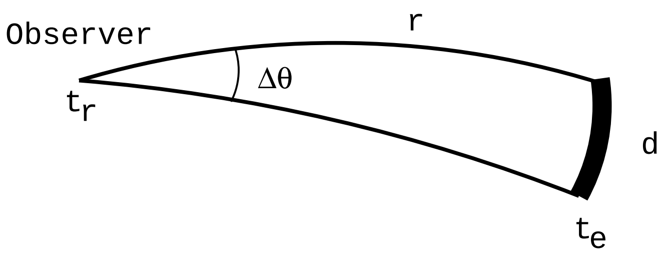

Figure 2: FRW geometry

Suppose that the light rays emitted at a time t_e are received by the observer at this instance, t_0. The proper size of the object at t_e is simply

$$d = a(t_e) r \Delta\theta.$$

On the other hand, the angular diameter distance D is *defined* in such a way that the relation

$$\Delta\theta D = d$$

holds, which yields the following expression for the angular diameter distance:

$$D = a(t_e) r.$$

The radial coordinate r can be calculated by noting that a light ray traces a null geodesic, so that the light rays emitted towards the observer obey the following equation:

$$c^2 \mathrm{d}t^2 = a(t)^2 \frac{\mathrm{d}r^2}{1 - kr^2}.$$

This leads to the following relations, depending on the specific geometry:

$$c \int_{t_e}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{1 - kr^2}} = \begin{cases} \sin^{-1} r & (k = 1) \\ r & (k = 0) \\ \sinh^{-1} r & (k = -1) \end{cases}.$$

At this point, it should be clear that to actually calculate angular diameter distances, we have to know the evolution of the distance scale $a(t)$.

3 Dependence on cosmological parameters

The evolution of the distance scale in the FRW metric, is described by the following equation:

$$\dot{a}(t)^2 - \frac{8\pi G}{3} \rho(t) a(t)^2 + kc^2 = 0.$$

Assuming that the total energy density ρ can be split into three parts: the energy density of the vacuum ρ_v, the energy density of matter ρ_m and the energy density of radiation ρ_r. The previous equation can then be rewritten as follows:

$$\dot{a}(t)^2 - \frac{8\pi G}{3} \left[\rho_v(t) + \rho_m(t) + \rho_r(t)\right] a(t)^2 + kc^2 = 0. \tag{1}$$

The adiabatic expansion of the universe, provides additional information:

$$\mathrm{d}\left(\rho a^3\right) = -\frac{p}{c^2} \mathrm{d}a^3$$

$$\Leftrightarrow \left(\rho + \frac{p}{c^2}\mathrm{d}a^3\right) + a^3\mathrm{d}\rho = 0.$$

For matter, the energy density is dominated by the mass of the matter and the pressure can be set to zero:

$$p_m = 0$$

Assuming that matter, radiation and vacuum energy density evolve independently, the following relations hold:

$$\frac{\mathrm{d}a^3}{a^3} = -\frac{\mathrm{d}\rho_m}{\rho_m}$$

$$\Leftrightarrow \ln a^3 = -\ln \rho_m + \text{constant}$$

$$\Leftrightarrow a^3\rho_m = \text{constant}$$

$$\Leftrightarrow a(t)^3\rho_m(t) = a_0^3\rho_{0m},$$

In which $a_0 = a(t_0)$, the current scale factor of the universe. This way, we find an expression for $\rho_m(t)$ in terms of $a(t)$:

$$\rho_m(t) = \left(\frac{a_0}{a(t)}\right)^3 \rho_{0m}.$$

For radiation, the equation of state is

$$p_r = \frac{1}{3}\rho_r c^2,$$

leading to the following relation:

$$\rho_r(t) = \left(\frac{a_0}{a(t)}\right)^4 \rho_{0r}.$$

If the energy density of the vacuum is constant (corresponding to a true cosmological *constant* Λ), the relation

$$\rho_v(t) = \rho_{0v}$$

holds, corresponding to the following equation of state:

$$p_v = -\rho_v c^2.$$

Using these relations, equation (1) can be rewritten as follows:

$$\dot{a}(t)^2 - \frac{8\pi G}{3}\left[\rho_{0m}\left(\frac{a_0}{a(t)}\right)^3 + \rho_{0r}\left(\frac{a_0}{a(t)}\right)^4 + \rho_{0v}\right] a(t)^2 + kc^2 = 0.$$

Introducing the critical energy density

$$\rho_c = \frac{3}{8\pi G}H_0^2$$

and writing

$$\Omega = \frac{\rho}{\rho_c},$$

this yields:

$$\dot{a}(t)^2 - H_0^2\left[\Omega_{0m}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_{0r}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_{0v}\right] a(t)^2 + \frac{kc^2}{H_0^2}H_0^2 = 0$$

Dividing the equation by a_0^2 results in the expression

$$\left(\frac{\dot{a}(t)}{a_0}\right)^2 - H_0^2\left[\Omega_{0m}\left(\frac{a_0}{a(t)}\right)^3 + \Omega_{0r}\left(\frac{a_0}{a(t)}\right)^4 + \Omega_{0v}\right]\left(\frac{a(t)}{a_0}\right)^2 + \frac{kc^2}{H_0^2 a_0^2}H_0^2 = 0,$$

which can be simplified by introducing

$$R(t) = \frac{a(t)}{a_0}$$

and

$$\Omega_{0k} = -\frac{kc^2}{H_0^2 a_0^2}.$$

This way, we obtain

$$\dot{R}(t)^2 - H_0^2\left[\Omega_{0m}R(t)^{-3} + \Omega_{0r}R(t)^{-4} + \Omega_{0v}\right]R(t)^2 - \Omega_{0k}H_0^2 = 0$$

$$\Leftrightarrow \dot{R}(t)^2 - H_0^2\left[\frac{\Omega_{0m}}{R(t)} + \frac{\Omega_{0r}}{R(t)^2} + \Omega_{0v}R(t)^2 + \Omega_{0k}\right] = 0.$$

Evaluating this expression at t_0, one gets the following relation:

$$\Omega_{0m} + \Omega_{0r} + \Omega_{0v} + \Omega_{0k} = 1,$$

since

$$R(t_0) = \frac{a(t_0)}{a_0} = 1$$

and

$$\dot{R}(t_0) = \frac{\dot{a}(t_0)}{a_0} = H_0.$$

Parametrizing our ignorance of H_0 by h:

$$H_0 = \frac{h}{T_H} \qquad \text{where } T_H = (100\ km\ s^{-1}\ Mpc^{-1})^{-1}$$

and introducing

$$T = \frac{t}{T_H},$$

the evolution of an FRW universe is described by:

$$\left(\frac{\mathrm{d}R}{\mathrm{d}T}\right)^2 - h^2\left[\frac{\Omega_{0m}}{R(T)} + \frac{\Omega_{0r}}{R(T)^2} + \Omega_{0v}R(T)^2 + \Omega_{0k}\right] = 0.$$

Previously, the vacuum energy density was assumed to be constant, but more generally we can write:

$$p_v = w\rho_v c^2.$$

It is an easy exercise to obtain the following expression when using this modified equation of state:

$$\left(\frac{\mathrm{d}R}{\mathrm{d}T}\right)^2 - h^2\left[\frac{\Omega_{0m}}{R(T)} + \frac{\Omega_{0r}}{R(T)^2} + \frac{\Omega_{0v}}{R(T)^{1+3w}} + \Omega_{0k}\right] = 0 \tag{2}$$

From this expression one easily sees that

$$\frac{\mathrm{d}T}{\mathrm{d}R} = \frac{1}{h\sqrt{\frac{\Omega_{0m}}{R} + \frac{\Omega_{0r}}{R^2} + \frac{\Omega_{0v}}{R^{1+3w}} + \Omega_{0k}}},$$

which can be used to rewrite the integral at the end of the previous section:

$$c\int_{t_e}^{t_0}\frac{\mathrm{d}t}{a(t)} = cT_H\int_{T_e}^{T_0}\frac{\mathrm{d}T}{a_0 R(T)} = c\frac{T_H}{a_0}\int_{R(T_e)}^{R(T_0)}\frac{\mathrm{d}R}{R}\frac{\mathrm{d}T}{\mathrm{d}R}$$

$$\Leftrightarrow c\int_{t_e}^{t_0}\frac{\mathrm{d}t}{a(t)} = c\frac{T_H}{a_0}\int_{R(T_e)}^{R(T_0)}\frac{\mathrm{d}R}{R}\frac{1}{h\sqrt{\frac{\Omega_{0m}}{R}+\frac{\Omega_{0r}}{R^2}+\frac{\Omega_{0v}}{R^{1+3w}}+\Omega_{0k}}}.$$

Noting that

$$R(T_e) = \frac{a(t_e)}{a_0} = \frac{1}{1+z}$$

where z is the observed redshift of the astronomical object, this can be written as

$$c\int_{t_e}^{t_0}\frac{\mathrm{d}t}{a(t)} = \frac{c}{h}\frac{T_H}{a_0}\int_{\frac{1}{1+z}}^{1}\frac{\mathrm{d}R}{R}\frac{1}{\sqrt{\frac{\Omega_{0m}}{R}+\frac{\Omega_{0r}}{R^2}+\frac{\Omega_{0v}}{R^{1+3w}}+\Omega_{0k}}}$$

$$\Leftrightarrow c\int_{t_e}^{t_0}\frac{\mathrm{d}t}{a(t)} = \frac{c}{h}\frac{T_H}{a_0}\int_{\frac{1}{1+z}}^{1}\frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R+\Omega_{0r}+\Omega_{0v}R^{1-3w}+\Omega_{0k}R^2}}. \qquad (3)$$

3.1 Flat space ($k = 0$)

In the $k = 0$ (and therefore $\Omega_0 k = 0$) case, the radial coordinate r is simply given by

$$r = c\int_{t_e}^{t_0}\frac{\mathrm{d}t}{a(t)},$$

from which one finds:

$$r = \frac{c}{h}\frac{T_H}{a_0}\int_{\frac{1}{1+z}}^{1}\frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R+\Omega_{0r}+\Omega_{0v}R^{1-3w}}}.$$

Substituting this into the expression of the angular diameter distance, we get:

$$D = \frac{cT_H}{h}\frac{a(t_e)}{a_0}\int_{\frac{1}{1+z}}^{1}\frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R+\Omega_{0r}+\Omega_{0v}R^{1-3w}}}$$

$$\Leftrightarrow D = \frac{1}{1+z}\frac{cT_H}{h}\int_{\frac{1}{1+z}}^{1}\frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R+\Omega_{0r}+\Omega_{0v}R^{1-3w}}}$$

3.2 Curved space ($k \neq 0$)

Using the definition of Ω_{k0}

$$\Omega_{0k} = -\frac{kc^2}{H_0^2 a_0^2},$$

we can express a_0 as follows:

$$a_0 = \frac{cT_H}{h}|\Omega_{0k}|^{-\frac{1}{2}}.$$

In the $k = +1$ case, the radial coordinate can be calculated in the following way:

$$\sin^{-1} r = \frac{c}{h}\frac{T_H h}{cT_H}\sqrt{-\Omega_{0k}}\int_{1}^{\frac{1}{1+z}}\frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R+\Omega_{0r}+\Omega_{0v}R^{1-3w}+\Omega_{0k}R^2}}$$

$$\Leftrightarrow \sin^{-1} r = \sqrt{-\Omega_{0k}} \int_1^{\frac{1}{1+z}} \frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w} + \Omega_{0k}R^2}}$$

$$\Leftrightarrow r = \sin\left[\sqrt{-\Omega_{0k}} \int_1^{\frac{1}{1+z}} \frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w} + \Omega_{0k}R^2}}\right].$$

In this case, the angular diameter distance is given by:

$$D = a(t_e)r = \frac{a(t_e)a_0}{a_0} r = \frac{1}{1+z}\frac{cT_H}{h\sqrt{-\Omega_{0k}}} \sin\left[\sqrt{-\Omega_{0k}} \int_{\frac{1}{1+z}}^{1} \frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w} + \Omega_{0k}R^2}}\right].$$

A similar calculation for the $k = -1$ case yields:

$$D = \frac{1}{1+z}\frac{cT_H}{h\sqrt{\Omega_{0k}}} \sinh\left[\sqrt{\Omega_{0k}} \int_{\frac{1}{1+z}}^{1} \frac{\mathrm{d}R}{\sqrt{\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w} + \Omega_{0k}R^2}}\right].$$

4 Summary

It is easy to generalize these results to obtain an expression for the angular diameter distance between objects at redshifts z_1 and z_2 ($z_1 < z_2$):

$$\boxed{D(z_1, z_2) = \begin{cases} \dfrac{1}{1+z_2}\dfrac{cT_H}{h\sqrt{-\Omega_{0k}}} \sin\left[\sqrt{-\Omega_{0k}} \displaystyle\int_{\frac{1}{1+z_2}}^{\frac{1}{1+z_1}} \mathrm{d}R \left(\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w} + \Omega_{0k}R^2\right)^{-\frac{1}{2}}\right] & (k = +1) \\[2ex] \dfrac{1}{1+z_2}\dfrac{cT_H}{h} \displaystyle\int_{\frac{1}{1+z_2}}^{\frac{1}{1+z_1}} \mathrm{d}R \left(\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w}\right)^{-\frac{1}{2}} & (k = 0) \\[2ex] \dfrac{1}{1+z_2}\dfrac{cT_H}{h\sqrt{\Omega_{0k}}} \sinh\left[\sqrt{\Omega_{0k}} \displaystyle\int_{\frac{1}{1+z_2}}^{\frac{1}{1+z_1}} \mathrm{d}R \left(\Omega_{0m}R + \Omega_{0r} + \Omega_{0v}R^{1-3w} + \Omega_{0k}R^2\right)^{-\frac{1}{2}}\right] & (k = -1). \end{cases}}$$